Non-Hermitian Interactions Between Harmonic Oscillators, with Applications to Stable, Lorentz-Violating QED

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Abstract

We examine a new application of the Holstein-Primakoff realization of the simple harmonic oscillator Hamiltonian. This involves the use of infinite-dimensional representations of the Lie algebra $su(2)$. The representations contain nonstandard raising and lowering operators, which are nonlinearly related to the standard $a^\dagger$ and a. The new operators also give rise to a natural family of two-oscillator couplings. These nonlinear couplings are not generally self-adjoint, but their low-energy limits are self-adjoint, exactly solvable, and stable. We discuss the structure of a theory involving these couplings. Such a theory might have as its ultra-low-energy limit a Lorentz-violating Abelian gauge theory, and we discuss the extremely strong astrophysical constraints on such a model.

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1   Introduction

The simple harmonic oscillator is one of the best understood systems in quantum mechanics, with applications in essentially all areas of physics. However, there may remain many interesting properties of this system that have not been fully understood or elucidated (see [1], for instance). In this paper, we present an example of this. This example provides a possible way for nonlocal interactions to stabilize a Lorentz-violating modification to the free photon sector of the standard model.

We shall show how the Holstein-Primakoff realization of the angular momentum Lie algebra $su(2)$ may be used in connection with the simple harmonic oscillator [2]. We first review the free oscillator, considered using the infinite-dimensional representations of this algebra. This leads naturally to a study of new nonlinear couplings between multiple identical harmonic oscillators. These couplings can be viewed in one of two ways. Taken at face value, the interactions are not self-adjoint, and the energy eigenvalues need not be real; however, the Hamiltonians in question do possess low-energy limits which are self-adjoint. The second possible viewpoint would be to take the formula which defines the real eigenvalues in the low-energy regime and extend that formula to cover the full range of the quantum numbers. This has the advantage of ensuring unitary evolution, but the additional complexity of the Hamiltonian is a corresponding disadvantage.

We shall describe how these new interactions may manifest themselves in a physical theory, with particular emphasis on couplings between the two polarization modes of the free electromagnetic field. However they are interpreted, the interactions we shall discuss may be relevant as part of a nonlocal, Lorentz-violating quantum field theory. Recently, there has been a great deal of interest in the possibility that Lorentz symmetry may not be exact in nature. A violation of this kind of fundamental symmetry could arise as part of the novel physics of the Planck scale. Relics of this violation would then persist even in the low-energy effective theory. The general local Lorentz-violating standard model extension (SME) has been developed [3, 4, 5], and the stability [6] and renormalizability [7] of this extension have been studied. Lorentz violation is a very interesting area of theoretical physics, because even superficially simple questions about its physics may have subtle and even ambiguous answers. For example, the study of the gauge invariance properties of and finite radiative corrections to Lorentz-violating field theories has proven to be a fruitful source for new theoretical insights [8, 9, 10, 11, 12, 13].

One significant difficulty with Lorentz-violating quantum field theories is that they frequently exhibit problems with stability [6, 14]. Yet it has been suggested that some form of nonlocality might overcome this problem [6, 15]. The harmonic oscillator interactions we consider can provide a concrete example for how this stabilization might work, if we adopt the second interpretation of these interactions as described above. In the limit of very low energies, the nonlocal interactions may couple together the two polarization modes of a free photon in exactly the same way as would a local, renormalizable operator from the SME [3, 14]. However, the nonlocality ensures that the theory remains stable,

even for very large photon numbers. Weak forms on nonlocality have also been considered in other Lorentz-violating contexts [16].

The kinds of infinite-dimensional representations of $su(2)$ that we will consider have also been introduced in the context of the Dirac Coulomb problem [17, 18] and in generalizations of the Dirac monopole [19], where they provide useful insights. Moreover, other Lie algebra also possess infinite-dimensional representations that are not representations of any corresponding Lie group. These representations might be useful in the study of certain quantum-mechanical systems, through a generalization of the techniques used in [17, 18, 19] or developed in this paper.

2   Single-Oscillator Operators

We shall begin by reviewing the Holstein-Primakoff realization of the harmonic oscillator raising and lowering algebra. Most frequently, when one studies the representations of $su(2)$ in connection with quantum mechanics, one is interested only in the finite-dimensional representations, which are countable and parameterized by the total angular momentum ℓ. One considers an operator $\vec{J} = (J_1, J_2, J_3) = \left(\frac{1}{2}J_+ + \frac{1}{2}J_-, \frac{1}{2i}J_+ - \frac{1}{2i}J_-, J_3\right)$, with standard commutation relations. Beginning from a highest weight state $|\ell\rangle$, with $J_+|\ell\rangle = 0$, one constructs each of the $2\ell + 1$ states $|\ell - s\rangle$ by acting on $|\ell\rangle$ with J_- s times and normalizing appropriately.

When 2ℓ is a nonnegative integer, there are only these $2\ell + 1$ states, because $J_-|-\ell\rangle = 0$. However, for more general values of the highest weight, the sequence of states does not terminate. Instead, one constructs an infinite tower of equally spaced states. This tower of states has a structure which is identical to that of the simple harmonic oscillator.

In fact, if we begin with a highest weight state $|\lambda\rangle$, with $J_3|\lambda\rangle = \lambda|\lambda\rangle$ (where 2λ is not a nonnegative integer), we may construct states $|\lambda - n\rangle$ for all nonnegative integers n, using $J_-|\lambda - n\rangle = \sqrt{\lambda(\lambda + 1) - (\lambda - n)(\lambda - n - 1)}|\lambda - n - 1\rangle$. Then the Hamiltonian $H_\lambda = -\omega J_3 + \omega\left(\lambda + \frac{1}{2}\right)$ has nondegenerate eigenvalues $\left(n + \frac{1}{2}\right)\omega$, which are precisely the energy eigenvalues of a harmonic oscillator (when we set $\hbar = 1$). Since a quantum-mechanical system is entirely specified by its Hilbert space and the Hamiltonian acting on that space, this is equivalent to an alternate description of the harmonic oscillator. (Note that this description is completely distinct from Schwinger's development of the angular momentum algebra in terms of harmonic oscillator states [20].)

We should point out that the state space is a representation only of the Lie algebra $su(2)$, not of the Lie group $SU(2)$. That this is the case should be clear from an examination of the spectrum of the Hamiltonian $H_{\hat{n}} = \omega\hat{n} \cdot \vec{J}$. This would be a Zeeman effect Hamiltonian if the representation of $\vec{J}$ were finite-dimensional, with $\vec{J}$ transforming in the adjoint representation of $SU(2)$. The $SU(2)$ symmetry would then dictate that the eigenenergies should be independent of the direction of $\hat{n}$. However, those energies are clearly not independent of $\hat{n}$ for the infinite-dimensional representations we are now

considering. Specifically, if $\hat{n} = -\hat{e}_3$, then we have a harmonic oscillator system, but if $\hat{n} = +\hat{e}_3$, then the energy is not bounded from below. Therefore, the infinite-dimensional operators cannot form a representation of the group $SU(2)$.

We shall now re-label our states, so that they match the usual harmonic oscillator nomenclature. We make the replacement $|\lambda - n\rangle \rightarrow |n\rangle$. Now, in addition to the standard harmonic oscillator raising and lowering operators $a^\dagger$ and a, we have a new set of raising and lowering operators J_- and J_+, given by

$$J_-|n\rangle = \sqrt{\lambda(\lambda+1) - (\lambda-n)(\lambda-n-1)}|n+1\rangle \tag{1}$$

$$J_+|n\rangle = \sqrt{\lambda(\lambda+1) - (\lambda-n)(\lambda-n+1)}|n-1\rangle, \tag{2}$$

or

$$J_- = a^\dagger\sqrt{2\lambda - a^\dagger a} \tag{3}$$

$$J_+ = a\sqrt{2\lambda - a^\dagger a + 1}. \tag{4}$$

The square root operators are to be interpreted as having eigenvalues equal to the square roots of the eigenvalues of the operators inside, with the same eigenvectors. For a mechanical oscillator, we may further express the $J_\pm$ in terms of the position and momentum operators x and p, using the usual linear relations connecting x and p to $a^\dagger$ and a; however, this substitution must be made with the understanding that the proper interpretation of the $J_\pm$ operators requires the use of the discrete "number of quanta present" basis of states.

The $J_\pm$ give us a new family of raising and lowering operators, parameterized by λ. These operators are distinct from the $a^\dagger$ and a for all finite λ. This is clear from the differing commutation relations $[a, a^\dagger] = 1$ and $[J_+, J_-] = 2J_3$. In general, for large n (large in comparison with λ and unity), the matrix elements of $J_\pm$ are larger that those of a and $a^\dagger$ by a factor of $\mathcal{O}(\sqrt{n})$. However, the new operators do include the a and $a^\dagger$ as limiting cases. As $\lambda \rightarrow \infty$, $\frac{J_+}{\sqrt{2\lambda}} \rightarrow a$ and $\frac{J_-}{\sqrt{2\lambda}} \rightarrow a^\dagger$. (These limits are to be interpreted in terms of the matrix elements of the operators involved, and all half-integral values of λ must be avoided as the limit is taken.)

We can see from (3) and (4) that $J_+ \neq J_-^\dagger$ for finite, non-half-integral λ, because the square roots in (3) and (4) may become imaginary. This means that J_1 and J_2 are not self-adjoint, a difficulty which we glossed over when we discussed the Hamiltonian $H_{\hat{n}}$. The Casimir operator $\vec{J}^2$ is self-adjoint, however. Moreover, in the basis of eigenstates of J_3, $J_+ = J_-^T$. Since the matrix elements of J_- and $J_+^\dagger$ differ only by phase factors in the J_3 basis, there is no ambiguity in defining the entire state space starting from the ground state. Finally, we point out that if λ is large and positive, then the non-self-adjoint character of $\vec{J}$ does not become apparent unless n is at least comparable to 2λ. These facts will prove important when we discuss the coupling between two harmonic oscillators.

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3   Multiple Coupled Oscillators

We shall now consider a novel application of this description of the harmonic oscillator. We may determine the energy eigenvalues exactly for certain systems of coupled identical oscillators in which the couplings are nonlinear. Let us consider the Hamiltonian

$$H \;=\; H_\lambda + H_\mu + g H_{\text{int}} \tag{5}$$

$$H \;=\; \left[-\omega J_{A3} + \left(\lambda + \frac{1}{2} \right) \omega \right] + \left[-\omega J_{B3} + \left(\mu + \frac{1}{2} \right) \omega \right] + g H_{\text{int}}. \tag{6}$$

$\vec{J}_A$ and $\vec{J}_B$ are two independent vectors of operators of the type we have been considering, corresponding to the highest weights λ and μ, respectively. H_{int} is an interaction, whose form we shall discuss shortly. This Hamiltonian has three adjustable parameters. λ and μ determine the structure of the harmonic oscillator representations that we are using. However, like g, they may be seen as parameters describing the interaction, because we have shifted the total energy in such a way as to make the spectrum of the free oscillators' Hamiltonian $H_\lambda + H_\mu$ independent of both λ and μ. These two parameters may be chosen freely, subject to the condition that neither 2λ nor 2μ is a nonnegative integer.

We shall choose an interaction $g H_{\text{int}}$ that is similar in form to $g \vec{J}_A \cdot \vec{J}_B$. If $g \neq 0$, $g \vec{J}_A \cdot \vec{J}_B$ is not self-adjoint; however, if λ and μ are large and positive, this problem will not be apparent in the vicinity of the ground state. So we take H_{int} to agree with $\vec{J}_A \cdot \vec{J}_B$ when $n_A + n_B + 1 < \min(2\lambda, 2\mu)$, where n_A and n_B are the principal quantum numbers of the two oscillators. We may then consider the effects of this interaction within this restricted ("low-energy") regime.

We may solve the *restricted* Hamiltonian using the ordinary techniques for the addition of angular momenta. The operator $\vec{J} = \vec{J}_A + \vec{J}_B$ has highest weights of the form $\lambda + \mu - i$ for all nonnegative integers i, and each value of i corresponds to a single irreducible component of the representation. The Clebsch-Gordon coefficients for these representations can be calculated by the standard method of applying lowering operators and using Gram-Schmidt orthonormalization. However, our primary interest is in the energy levels.

The eigenvalues of $H_\lambda + H_\mu$ are just $(n_A + n_B + 1)\omega$. When we change the basis, to use the "total angular momentum" $\vec{J}$, this part of the Hamiltonian becomes

$$H_\lambda + H_\mu = -\omega J_3 + (\lambda + \mu + 1)\omega. \tag{7}$$

The eigenstates of the system are parameterized by the highest weight $\lambda + \mu - i$ and by the "number of quanta present" (that is, the number of applications of J_- on the highest weight state required to produce a given state), n. If we denote these states by $|i, n\rangle$, then it is clear that $(H_\lambda + H_\mu)|i, n\rangle = (i + n + 1)\omega |i, n\rangle$. Since i and n have the same range as n_A and n_B (all must be nonnegative integers), this verifies that the free system has the same spectrum in each basis.

The coupling term must be calculated in the $\vec{J}$ basis. This is easily done, using $\vec{J}_A \cdot \vec{J}_B = \frac{1}{2}\left(\vec{J}^2 - \vec{J}_A^2 - \vec{J}_B^2\right)$. For a state of specified i, $\vec{J}_A \cdot \vec{J}_B$ has the eigenvalue $\lambda\mu - \left(\lambda + \mu + \frac{1}{2}\right)i + \frac{1}{2}i^2$. So the total energy is

$$E_{i,n} = \omega(i + n + 1) + g\left[\lambda\mu - \left(\lambda + \mu + \frac{1}{2}\right)i + \frac{1}{2}i^2\right]. \tag{8}$$

This formula holds exactly in the entire low-energy subspace.

We must now turn our attention to the general definition of H_{int}. There are two natural ways to define this interaction. The first possibility is that $H_{\text{int}} = \vec{J}_A \cdot \vec{J}_B$ exactly, and the fundamental Hamiltonian is not self-adjoint. The second possibility involves a less drastic modification of the structure of the theory. We simply take the exact Hamiltonian to be defined by its eigenvalues, which have the form (8). There is a complete set of states $\{|i, n\rangle\}$ corresponding to these eigenenergies, and for $i + n + 1 < \min(2\lambda, 2\mu)$, these states agree with the ones found above. The new Hilbert space is not equivalent to the old, two-oscillator Hilbert space, but the restricted, low-energy subspaces of these two Hilbert spaces are isomorphic. We shall henceforth adopt this second definition of H_{int} (although most of our statements will concern only the low-energy subspace, where the two definitions are equivalent).

A few words about the various parameters are now in order. Our coupled system is stable only if $g \geq 0$; if $g < 0$, the energy is not bounded below. We also see that the structure of the energy levels depends only on g and $\lambda + \mu$; the separate values of λ and μ only affect the zero-point energy.

This is a strongly nonlinearly coupled system. We may recover the more usual result for the spectrum in the presence of a linear coupling between the oscillators by setting $\lambda = \mu$ and expressing $a^\dagger$ and a in terms of the $\lambda \to \infty$ limits of J_+ and J_-. (Note that as $\lambda \to \infty$, there are no problems with the operators being self-adjoint.) The interaction $g\left(a_A a_B^\dagger + a_A^\dagger a_B\right)$ is the limit of $\frac{g}{\lambda}\left(\vec{J}_A \cdot \vec{J}_B - J_{A3}J_{B3}\right)$ as λ approaches infinity. The two terms $\frac{g}{\lambda}\vec{J}_A \cdot \vec{J}_B$ and $\frac{g}{\lambda}J_{A3}J_{B3}$ each commute with the noninteracting Hamiltonian; and, although they do not commute with each other for finite values of λ, they do commute in the infinite limit. We may see this by evaluating the two operators for finite λ in different bases. In the $\vec{J}$ basis, the first term is diagonal, with eigenvalues $g\left[\lambda - \left(2 + \frac{1}{2\lambda}\right)i + \frac{1}{2\lambda}i^2\right]$, just as calculated above. The second term is diagonal in the J_A, J_B basis, with eigenvalues $\frac{g}{\lambda}(\lambda - n_A)(\lambda - n_B)$. As $\lambda \to \infty$, these eigenvalues become $g(\lambda - 2i)$ and $g(\lambda - n_A - n_B)$, respectively. However, as we saw when we discussed the noninteracting case, the total number operator $n_A + n_B = n + i$ is diagonal in both bases, so $g(\lambda - n_A - n_B) = g(\lambda - n - i)$. Then the total energy shift, which is now the difference between $g(\lambda - 2i)$ and $g(\lambda - n - i)$, becomes simply $g(n - i)$. That is, we have two decoupled oscillators with frequencies $\omega \pm g$, which is just the usual result.

More general interactions are also possible. We may replace $g\vec{J}_A \cdot \vec{J}_B$ with an arbitrary function of $\vec{J}_A \cdot \vec{J}_B$. Any such interaction will still commute with $H_\lambda + H_\mu$, and all the

same considerations will still apply. Other generalizations are possible as well, through the use of other well-known properties of $su(2)$. For example, we may generalize to the coupling of N identical oscillators, with the identity

$$\sum_{1 \leq C < D \leq N} \vec{J}_C \cdot \vec{J}_D = \frac{1}{2} \left[\left(\sum_{C=1}^{N} \vec{J}_C \right)^2 - \sum_{C=1}^{N} \left(\vec{J}_C \right)^2 \right]. \tag{9}$$

However, it is important to keep in mind that many approximations that are typically used when one studies more than two interacting angular momenta will break down when working with infinite-dimensional representations of $su(2)$.

4 Application to Lorentz-Violating QED

One of the simplest situations in which pairs of identical harmonic oscillators arise is in an Abelian gauge theory in 3+1 dimensions. An interaction of the form (6) might be relevant as a modification of the photon sector of quantum electrodynamics. For the gauge sector alone, this interaction could be introduced separately at each value of the photon momentum. The selection of a specific basis of polarization states for each momentum, and the assignment of the couplings λ, μ, and g generally breaks Lorentz symmetry and may also break parity invariance. Moreover, the operators $J_\pm$ are nonlocal, since they involve the total energy present in a given mode of the electromagnetic field, and it is not possible to express this sort of interaction conveniently in terms of the ordinary electromagnetic field operators A^μ and $F^{\mu\nu}$. However, if we are willing to allow these modifications to the structure of the theory, the interaction is (in the absence of charges) exactly solvable. Since there is current interest in exotic modifications of QED, this type of interaction may be worthy of further investigation.

We must also say a word about the gauge invariance of this QED modification. The $\vec{J}_A \cdot \vec{J}_B$ interaction is formulated in terms of creation and annihilation operators (i.e. in the canonical quantization formalism), and so any discussion of gauge invariance will necessarily be complicated by the difficulties that are associated with the canonical quantization of gauge fields. However, at low energies, the interactions we have introduced are clearly consistent with gauge invariance in the following sense. If we specialize to the Coulomb gauge and quantize the transverse modes of the theory, then the low-energy interaction may be introduced without difficulty. It does not affect the number of polarization states (and is, in fact, strongly dependent upon this number), so it does not spoil gauge invariance in this fashion. However, it is possible that it may damage gauge invariance at higher energies or when interactions with matter are considered.

Although the interactions we are considering cannot be expressed simply in terms of the standard electromagnetic field operators, the ultra-low-energy limit of our theory could well be expressible in such a form. We shall shortly show that this is indeed the case for a particular class of models. The embedding of a ultra-low-energy Lorentz-violating effective

field theory within the framework of our fundamental theory is attractive for several reasons. First, the interactions we have considered have eigenenergies that are exactly known. Second, although Lorentz-violating theories may exhibit stability problems, our theory does not; the i^2 term in (8) ensures this. This example shows that there can exist stable nonlocal interactions which have local, Lorentz-violating Lagrangian theories as their ultra-low-energy limits. Third, such an embedding demonstrates the intriguing possibility that our conventional basis of polarization states many be inadequate for the description of the full Hilbert space of a more fundamental theory.

We shall therefore examine the ultra-low-energy behavior of our QED modification, to see to what sort of effective theory it may correspond. Since the energy-level differences depend only on $\lambda + \mu$, we shall set $\lambda = \mu$. We take λ to be a very large number, large in comparison with any relevant photon occupation number. (This is what we mean by "ultra-low-energy.") We may then neglect $a^\dagger a$ compared to λ in (3) and (4). It immediately follows that the energy eigenstates are approximately given by

$$|i, n\rangle \approx \frac{\left(a_A^\dagger - a_B^\dagger\right)^i \left(a_A^\dagger + a_B^\dagger\right)^n}{\sqrt{2^{i+n}(i!)(n!)}}|0, 0\rangle. \tag{10}$$

Deviations from this expression are suppressed by a factor of $\mathcal{O}(\lambda^{-1/2})$. The corresponding energies, in the same approximation, are

$$E_{i,n} \approx (n + i)\omega + 2\lambda gi, \tag{11}$$

where we have dropped the zero-point energy.

These results imply that the polarization modes corresponding to $a_+^\dagger \equiv \frac{1}{\sqrt{2}}\left(a_A^\dagger + a_B^\dagger\right)$ and $a_-^\dagger \equiv \frac{1}{\sqrt{2}}\left(a_A^\dagger - a_B^\dagger\right)$ have different frequencies. The two frequencies are shifted from their mean value by $\pm\frac{\Delta\omega}{2} \approx \pm g\lambda$, and the effective Hamiltonian is

$$H_{\text{eff}} = (\omega + 2\lambda g)a_-^\dagger a_- + \omega a_+^\dagger a_+. \tag{12}$$

This same effective Hamiltonian arises naturally in the context of a CPT-even, Lorentz-violating modification of the photon sector. For a theory with Lagrange density

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{4}k_{\kappa\tau\mu\nu}F^{\kappa\tau}F^{\mu\nu} \tag{13}$$

(where $k_{\kappa\tau\mu\nu}$ has the symmetries of a Riemann tensor and is double traceless), the expressions for the photon modes' frequencies are (to leading order in $k_{\kappa\tau\mu\nu}$) [6, 7, 11]

$$\omega_\pm = (1 + \rho \pm \sigma)|\vec{p}|. \tag{14}$$

Here, $\vec{p}$ is the photons' 3-momentum, $\rho = -\frac{1}{2}\tilde{k}_\alpha{}^\alpha$, and $\sigma^2 = \frac{1}{2}\tilde{k}_{\alpha\beta}\tilde{k}^{\alpha\beta} - \rho^2$, with $\tilde{k}_{\alpha\beta} = k_{\alpha\mu\beta\nu}\hat{p}^\mu\hat{p}^\nu$ and $\hat{p}^\mu = (1, \vec{p}/|\vec{p}|)$. The approximate frequencies given by (14) and (12) correspond if $\rho = \sigma = g\lambda$. So any theory with $\rho = \sigma \geq 0$ for all $\vec{p}$ will reproduce the entire

low-energy behavior of our modified theory. Theories with this property indeed exist; for example, if $k_{\kappa\tau\mu\nu}$ has the form

$$k_{\kappa\tau\mu\nu} = (v_\kappa u_\tau - v_\tau u_\kappa)(v_\mu u_\nu - v_\nu u_\mu) - \frac{1}{6}\left[v^2 u^2 - (v \cdot u)^2\right](g_{\kappa\mu}g_{\tau\nu} - g_{\kappa\tau}g_{\mu\nu}) + (\kappa \leftrightarrow \mu),$$
(15)

then $\rho = \sigma = -w^2$, where $w^\mu = v^\mu(u \cdot \hat{p}) - u^\mu(v \cdot \hat{p})$. If $v^\mu = \left(V, \vec{0}\right)$ and $u^\mu = (0, \vec{u})$, with $|\vec{u}| = 1$, then $\rho = V^2 \sin^2\theta$, where θ is the angle between $\vec{u}$ and $\vec{p}$. This particular model is parity-preserving, with three independent parameters.

Moreover, *any* theory with nonvanishing σ will demonstrate a splitting between polarization modes, as would arise in our $su(2)$-modified QED. There is therefore a large theoretical parameter space in which the theory given by (13) can be embedded in a $su(2)$ coupling model.

These embeddings all require that, for a fixed direction $\hat{p}$, the coupling $g\lambda$ must be proportional to $|\vec{p}|$. It would seem most natural for λ, which represents the number of photons that must be present in a mode of the electromagnetic field in order for the failure of the polarization state basis to be apparent, to remain large for all values of $|\vec{p}|$. We therefore speculate that λ may be a $|\vec{p}|$-independent (or $\vec{p}$-independent) constant, while g scales with the magnitude of $\vec{p}$.

Based upon astrophysical experiments, the physical value of σ is strongly constrained, to parts in 10^{31} or better [21, 22]. This represents an even stronger constraint on the $su(2)$ model, because λ is necessarily very large. It follows that $g = \sigma/\lambda$ is correspondingly smaller. Our modification of QED is thus physically reasonable only in a very, very small region of parameter space.

5   Conclusions

In summary, we have studied an applicaiotn of the Holstein-Primakoff realization of the simple harmonic oscillator operator algebra. This has involved the introduction of a set of raising and lowering operators that obey the angular momentum commutation relations. The properties of these operators allow us to solve exactly for the low-energy behavior of a theory with a particular nonlinear interaction. The high-energy extension of this model may involve either a non-self-adjoint Hamiltonian or a new basis of states.

This model has an interesting application in Lorentz-violating physics, where stability is typically a problem. The ultra-low-energy limit of the interaction we have considered resembles the effect of a small Lorentz-violating correction to free QED. At higher energies, the nonlocality of the special harmonic oscillator interactions we are considering can stabilize the Lorentz-violating theory. Physically, these kinds of effects are extremely strongly constrained by astronomical observations. However, this theory provides an useful insight into how nonlocality and Lorentz violation may combine to form a well-behaved theory.

Acknowledgments

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